

A₁-REGULARITY AND BOUNDEDNESS OF CALDERÓN-ZYGMUND OPERATORS. II

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ABSTRACT. Proof is given for the “only if” part of the result stated in the previous paper of the series that a suitably nondegenerate Calderón-Zygmund operator T is bounded in a Banach lattice X on \mathbb{R}^n if and only if the Hardy-Littlewood maximal operator M is bounded in both X and X' , under the assumption that X has the Fatou property and X is p -convex and q -concave with some $1 < p, q < \infty$. We also get rid of an application of a fixed point theorem in the proof of the main lemma and give an improved version of an earlier result concerning the divisibility of BMO-regularity.

This paper is closely related to [7] and contains essentially no new non-technical results, hence for the background and the generalities we refer the reader to [7].

A. Yu. Karlovich and L. Maligranda kindly pointed out to the author that the proof of [7, Theorem 16] has a flaw, namely that the relationship $(XL_s)' = X'L_{s'}$ is incorrect (and in fact it is always false). Unfortunately, it is not clear if [7, Theorem 16] is true in the stated form.

Nevertheless, we will see that the main result of [7] is still true with only a slight loss of generality concerning the nondegeneracy assumption imposed on a Calderón-Zygmund operator T . Specifically, in place of A₂-nondegeneracy of T (which is a condition that the boundedness of T in $L_2(w^{-\frac{1}{2}})$ implies $w \in A_2$ with an estimate for the constant) we require that the kernels of both T and its conjugate T^* satisfy a standard assumption on growth along a certain singular direction (see [10, Chapter 5, §4.6]).

Definition 1. *We say that a singular integral operator T on \mathbb{R}^n is nondegenerate if there exists a constant $c > 0$ and some $x_0 \in \mathbb{R}^n \setminus \{0\}$ such that for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and any locally summable nonnegative function f supported on B we have*

$$(1) \quad |Tf(x)| \geq c \frac{1}{|B|} \int_B f$$

for all $x \in B \pm rx_0$.

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For example, the Hilbert transform H and any of the Riesz transforms R_j are nondegenerate in this sense. A nondegenerate operator T is also A_2 -nondegenerate. For details see [10, Chapter 5, §4.6].

Theorem 2. *Suppose that X is a Banach lattice of measurable functions on $\mathbb{R}^n \times \Omega$ that satisfies the Fatou property and X is p -convex and q -concave with some $1 < p, q < \infty$. Let T be a Calderón-Zygmund operator in $L_2(\mathbb{R}^n)$ such that both T and T^* are nondegenerate. The following conditions are equivalent.*

- (1) *The Hardy-Littlewood maximal operator M acts boundedly in X and in the order dual X' of X .*
- (2) *All Calderón-Zygmund operators act boundedly in X .*
- (3) *T acts boundedly in X .*

Thus, concerning the necessity of A_1 -regularity we make no claims about the general spaces of homogeneous type, although in many cases a suitable generalization of Definition 1 seems to be possible. Another subtle loss of generality is that in contrast to [10, Theorem 16] in the proof of $3 \Rightarrow 1$ we take advantage of the assumption that T is a Calderón-Zygmund operator as well as a nondegenerate operator, specifically that T is bounded in L_t for $1 < t < \infty$ with norm $O(t)$ as $t \rightarrow \infty$.

For the proof of $1 \Rightarrow 2 \Rightarrow 3$ of Theorem 2 see [7]. The proof of $3 \Rightarrow 1$ essentially follows the scheme of the flawed proof of [7, Theorem 16], but it seems to require a much more delicate approach that we will present throughout the rest of the paper, leading to the proof itself given at the end of Section 5 below. We briefly outline the structure of the argument, the details of which are also of some independent interest.

The following result was established (with some caveats) in [5, Theorem A']; a complete proof in the stated form can be found in [6, Theorem 4]. Here and elsewhere (S, ν) is a space of homogeneous type and (Ω, μ) is a σ -finite measurable space.

Theorem 3. *Suppose that Y is a Banach lattice on $(S \times \Omega, \nu \times \mu)$ with an order continuous norm. If a linear operator T is bounded in $Y^{\frac{1}{2}}$ then for every $f \in Y'$ there exists a majorant $w \geq |f|$, $\|w\|_{Y'} \leq 2\|f\|_{Y'}$, such that $\|T\|_{L_2(w^{-\frac{1}{2}}) \rightarrow L_2(w^{-\frac{1}{2}})} \leq C\|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}}$, where C depends only on the Grothendieck constant K_G .*

This yields almost at once the following version of Theorem 2 that we will need in the proof of Theorem 2, showing that Theorem 2 is also valid for $p = 2$, $q = \infty$ (and by duality for $p = 1$ and $q = 2$), provided that X

(respectively, X') has order continuous norm. The proof is given in Section 1 below.

Theorem 4. *Suppose that X is a 2-convex Banach lattice of measurable functions on $(S \times \Omega, \nu \times \mu)$ having order continuous norm and the Fatou property. Let T be an A_2 -nondegenerate linear operator in $L_2(S \times \Omega)$. If T acts boundedly in X then the maximal operator M is bounded in both X and X' with a suitable estimate for the constants.*

In contrast to [7], in the present work we use the standard definition of the constant $[w]_{A_p}$, $p > 1$ of a Muckenhoupt weight $w \in A_p$ on $(S \times \Omega, \nu \times \mu)$ based on the Muckenhoupt condition:

$$[w]_{A_p} = \operatorname{ess\,sup}_{\omega \in \Omega} \sup_B \left(\frac{1}{\nu(B)} \int_B w(\cdot, \omega) \right) \left(\frac{1}{\nu(B)} \int_B w^{-\frac{1}{p-1}}(\cdot, \omega) \right)^{p-1},$$

where the supremum is taken over all balls $B \subset S$.

Recall that a quasi-normed lattice X is called A_p regular with constants (C, m) if every $f \in X$ admits a majorant $w \in X$, $w \geq |f|$, such that $\|w\|_X \leq m\|f\|_X$ and w belongs to the Muckenhoupt class A_p with $[w]_{A_p} \leq C$.

In Section 2 we give (Proposition 7) a simplified proof of [7, Proposition 8] that does not use a fixed point theorem. This yields a slightly improved version (Proposition 8) of [7, Proposition 12] stating that A_∞ -regularity of both X and X' implies A_1 -regularity of these lattices, where the assumption that X satisfies the Fatou property is replaced by a weaker assumption that X' is a norming lattice for X . Thus it suffices to establish that condition 3 of Theorem 2 implies that X' is A_∞ -regular; interchanging X with X' would then show that X is also A_∞ -regular.

Under condition 3 of Theorem 4 we may apply Theorem 3 to lattice $Y = X^r L_s$ with some fixed $r > 1$ sufficiently close to 1 and all sufficiently large s , since T is bounded in $Y^{\frac{1}{2}}$ by interpolation with some estimate for the norm that grows with s . This yields A_2 -regularity of $Y' = (X^{rs'})'_{\frac{1}{s'}}$, with an estimate on the growth of the constant C_s as $s \rightarrow \infty$. Now the key idea is to show that the A_2 -majorants w of functions from Y' also satisfy the reverse Hölder inequality with exponent s' for some sufficiently large s , which would yield A_2 -regularity of $(X^u)'$, $u = rs'$, and thus the required A_2 -regularity of the lattice $X' = (X^u)'_{\frac{1}{u}} L_1^{1-\frac{1}{u}}$.

However, as discussed in Section 4 below, in order to get an estimate for C_s with a suitable rate of growth we also need to make sure that the weight w appearing in the conclusion of Theorem 3 (applied to Y) satisfies some additional assumptions, namely that w^{-1} is a doubling weight with

a constant independent of s . Theorem 4 allows us to obtain A_1 -regularity of $Y'^{\frac{1}{2}}L_t^{\frac{1}{2}}$ from condition 3 of Theorem 2 with a sufficiently large fixed t , where an estimate for the constants is independent of s . An extension (Theorem 15) of [6, Theorem 2] concerning the divisibility of A_p -regularity, which we introduce in Section 3 below, allows us to prove that Y' admits suitable majorants w such that $w^{-1} \in A_3$ (and hence w^{-1} is a doubling weight) with a constant independent of s , and an adaptation (Theorem 18) of the original fixed point argument from [7, §2] makes it possible to impose this condition on the weights appearing in the conclusion of Theorem 3, thus completing the proof.

1. PROOF OF THEOREM 4

Suppose that X and Y are normed lattices on a measurable space Ω . Lattice Y is said to be *norming* for X if $fg \in L_1$ for all $f \in X$ and $g \in Y$ and $\|f\|_X = \sup_{g \in Y, \|g\|_Y=1} \int_{\Omega} |fg|$ for all $f \in X$. A normed lattice X is always norming for its order dual X' . Conversely, it is well known that X' is a norming lattice for X if X satisfies either the Fatou property (implying that $X = X''$), or if X is a Banach lattice having order continuous norm (since then $X' = X^*$). The fact that $w \in A_p$ if and only if the maximal operator M is bounded in $L_p(w^{-\frac{1}{p}})$ with the appropriate estimates of the constants yields at once the following result; see [6, Proposition 13].

Proposition 5. *Suppose that X and Y are normed lattices on $(S \times \Omega, \nu \times \mu)$ such that Y is a norming space for X . If Y is A_p -regular with some $p > 1$ then $X^{\frac{1}{p}}$ is A_1 -regular with appropriate estimates for the constants.*

The following result is a particular case of [8, Proposition 13]; we give a complete proof for clarity.

Proposition 6. *Suppose that Z is an A_2 -regular quasi-normed lattice on $(S \times \Omega, \nu \times \mu)$. Then lattice $Z^{\frac{1}{2}}L_1^{\frac{1}{2}}$ is A_1 -regular.*

Indeed, suppose that $f \in Z^{\frac{1}{2}}L_1^{\frac{1}{2}} = Z^{\frac{1}{2}}L_2$ with norm 1, so there exist some $g \in Z$ and $h \in L_2$ with norms at most 2 such that $g \geq 0$ almost everywhere and $f = g^{\frac{1}{2}}h$. Let w be a suitable A_2 -majorant for g in Z . Then

we also have $w^{-1} \in A_2$ and

$$\begin{aligned} \|Mf\|_{Z^{\frac{1}{2}}L_2} &\leq \left\| w^{\frac{1}{2}} \right\|_{Z^{\frac{1}{2}}} \left\| (Mf)w^{-\frac{1}{2}} \right\|_{L_2} = \|w\|_Z^{\frac{1}{2}} \left\| (Mf)w^{-\frac{1}{2}} \right\|_{L_2} \leq \\ &c \left\| (Mf)w^{-\frac{1}{2}} \right\|_{L_2} = c \|Mf\|_{L_2(w^{\frac{1}{2}})} \leq \\ c' \|f\|_{L_2(w^{\frac{1}{2}})} &= c' \left\| fw^{-\frac{1}{2}} \right\|_{L_2} = c' \left\| \left(\frac{g}{w} \right)^{\frac{1}{2}} h \right\|_{L_2} \leq c' \|h\|_{L_2} \leq c'' \end{aligned}$$

with some suitable constants c , c' and c'' , so M is bounded in the lattice $Z^{\frac{1}{2}}L_1^{\frac{1}{2}}$ which is thus A_1 -regular as claimed.

Now we can prove Theorem 4. Since X is 2-convex, we may apply Theorem 3 to lattice $Y = X^2$ and obtain A_2 -regularity of lattice Y' by the assumed A_2 -nondegeneracy of operator T . By Proposition 5 lattice $Y^{\frac{1}{2}} = X$ is then A_1 -regular, and by Proposition 6 lattice $Y'^{\frac{1}{2}}L_1^{\frac{1}{2}} = Y'^{\frac{1}{2}}L_{\infty}'^{\frac{1}{2}} = \left(Y^{\frac{1}{2}}L_{\infty}^{\frac{1}{2}}\right)' = X'$ is also A_1 -regular as claimed.

2. MAIN LEMMA REVISITED

Recall that a lattice X is called A_p -regular if functions from X admit A_p majorants with the appropriate control on the norm; see also Definition 10 in Section 3 below. Lattice X is A_{∞} -regular if and only if it is A_p -regular with some $p < \infty$. A_1 -regularity is equivalent to the boundedness of the Hardy-Littlewood maximal operator (see, e. g., [6, Proposition 1]).

The following result was established in [7, Theorem 8] with the help of a fixed point theorem under an additional assumption that X is a Banach lattice satisfying the Fatou property. However, we will now see that for the proof it suffices to carry out a slightly modified version of estimate [7, (6)] with the appropriate majorants.

Proposition 7. *Suppose that X is a quasi-Banach lattice of measurable functions on $(S \times \Omega, \mu \times \nu)$ such that X is A_p -regular with some $1 \leq p < \infty$ and X^{δ} is A_1 -regular with some $\delta > 0$. Then X is A_1 -regular with an appropriate estimate for the constants depending only on the corresponding A_p -regularity constants of X , A_1 -regularity constants of X^{δ} and the value of δ .*

Indeed, let $f \in X$. Then there exists an A_p -majorant w for f in X , and in turn there exists an A_1 -majorant u for w^{δ} in X^{δ} . We fix some $\omega \in \Omega$ such that $w(\cdot, \omega) \in A_p$ and $u(\cdot, \omega) \in A_1$, and let $B(x, r) \subset S$, $x \in S$, $r > 0$, be an arbitrary ball in S . Sequential application of the A_p condition satisfied by weight w , the Jensen inequality with convex function $t \mapsto t^{-\delta(p-1)}$, $t > 0$,

and the A_1 condition satisfied by the weight u yield

$$\begin{aligned}
(2) \quad & \frac{1}{\nu(B(x, r))} \int_{B(x, r)} |f(\cdot, \omega)| \leq \frac{1}{\nu(B(x, r))} \int_{B(x, r)} w(\cdot, \omega) \leq \\
& c \left[\frac{1}{\nu(B(x, r))} \int_{B(x, r)} [w(\cdot, \omega)]^{-\frac{1}{p-1}} \right]^{-(p-1)} = \\
& c \left[\frac{1}{\nu(B(x, r))} \int_{B(x, r)} [w(\cdot, \omega)]^{-\frac{1}{p-1}} \right]^{-\delta(p-1) \cdot \frac{1}{\delta}} \leq \\
& c \left[\frac{1}{\nu(B(x, r))} \int_{B(x, r)} [w(\cdot, \omega)]^\delta \right]^{\frac{1}{\delta}} \leq \\
& c \left[\frac{1}{\nu(B(x, r))} \int_{B(x, r)} u(\cdot, \omega) \right]^{\frac{1}{\delta}} \leq c' [u(x, \omega)]^{\frac{1}{\delta}}
\end{aligned}$$

for almost all $x \in S$ with suitable constants c and c' . Since ω , x and B are arbitrary, (2) implies that $Mf \leq c'u^{\frac{1}{\delta}}$ almost everywhere, so $\|Mf\|_X \leq c'\|u\|_{X^\delta}^{\frac{1}{\delta}} \leq c''\|w^\delta\|_{X^\delta}^{\frac{1}{\delta}} = c''\|w\|_X \leq c'''\|f\|_X$ with some appropriate constants c'' and c''' . Thus M is bounded in X with an appropriate estimate of the norm, and so lattice X is suitably A_1 -regular.

Proposition 8. *Let X be a normed lattice on $(S \times \Omega, \nu \times \mu)$ such that X' is norming for X . Suppose that both X and X' are A_∞ -regular. Then both X and X' are A_1 -regular.*

Indeed, since X and X' are A_∞ -regular, they are also A_p -regular with some $p < \infty$. By Proposition 5 both $X'^{\frac{1}{p}}$ and $X^{\frac{1}{p}}$ are then A_1 -regular, and it remains to apply Theorem 7 to X and to X' with $\delta = \frac{1}{p}$.

3. DIVISIBILITY OF A_p -REGULARITY

It is often convenient to think about Muckenhoupt weights in terms of the Jones factorization theorem (see, e. g., [10, Chapter 5, §5.3]: $w \in A_p$ if and only if $w = \omega_0 \omega_1^{1-p}$ with some weights $\omega_0, \omega_1 \in A_1$ with the appropriate estimates on the constants. This makes it intuitive that, for example, division by the A_1 weights turns A_p weights into A_{p+1} weights, which is the main insight behind the divisibility theorem for A_p -regularity [6, Theorem 2]: under certain assumptions on Banach lattices X and Y , if lattice XY is A_p -regular and lattice Y is A_1 -regular then lattice X is A_{p+1} -regular.

However, in the present work a somewhat more general problem arises: we need to make sure that a lattice X admits majorants w such that $w^{-1} \in$

A_∞ based on the assumption that lattice $(XY)^\delta$ is A_1 -regular with an A_1 -regular lattice Y and some $\delta > 0$. With that in mind we introduce the following notions; see also [8, §1].

Definition 9. Let $\alpha, \beta \geq 0$. We say that a weight w on $(S \times \Omega, \nu \times \mu)$ belongs to class F_β^α with a constant C if there exist two weights $\omega_0, \omega_1 \in A_1$ with constant C such that $w = \frac{\omega_0^\alpha}{\omega_1^\beta}$.

Definition 10. Let $\alpha, \beta \geq 0$, and suppose that X is a quasi-normed lattice on $(S \times \Omega, \nu \times \mu)$. We say that X is F_β^α -regular with constants (C, m) if for any $f \in X$ there exists a majorant $w \in X$, $w \geq |f|$ such that $\|w\|_X \leq m\|f\|_X$ and $w \in F_\beta^\alpha$ with constant C .

“F” in the notation F_β^α stands for “factorizable weight”, and the properties of the A_1 weights imply that at least in the local terms ω_0 roughly represents the “poles” of the weight w where the weight takes relatively large values, whereas ω_1 represents the “zeroes” of w where the weight is relatively small. The corresponding factorization is generally not unique.

Since $\omega \in A_1$ implies $\omega^\delta \in A_1$ for $0 < \delta \leq 1$, we see that $F_\beta^\alpha \subset F_{\beta_1}^{\alpha_1}$ for $\alpha \leq \alpha_1$ and $\beta \leq \beta_1$. Likewise, F_β^α -regularity of a lattice X implies its $F_{\beta_1}^{\alpha_1}$ -regularity.

It is easy to see that these properties are closely related to A_p -regularity.

Proposition 11. Suppose that $\alpha > 0$, $\beta \geq 0$ and w is a weight on $(S \times \Omega, \nu \times \mu)$. Then $w \in F_\beta^\alpha$ if and only if $w^{\frac{1}{\alpha}} \in A_{\frac{\beta}{\alpha}+1}$, and $w \in F_\beta^0$ if and only if $w^{-\frac{1}{\beta}} \in A_1$ with the appropriate estimates on the constants.

Indeed, it suffices to observe that $\frac{\omega_0^\alpha}{\omega_1^\beta} = \left(\omega_0 \omega_1^{-\frac{\beta}{\alpha}}\right)^\alpha$ for $\alpha > 0$.

Proposition 11 yields at once the corresponding result for F_β^α -regularity.

Proposition 12. Let X be a quasi-normed lattice on $(S \times \Omega, \nu \times \mu)$, and suppose that $\alpha > 0$, $\beta \geq 0$. Lattice X is F_β^α -regular if and only if lattice $X^{\frac{1}{\alpha}}$ is $A_{\frac{\beta}{\alpha}+1}$ -regular.

Incidentally, as a corollary we get yet another characterization of the property $\log w \in \text{BMO}$ and the corresponding BMO-regularity in terms of $w \in F_\beta^\alpha$ with some α and β and, respectively, F_β^α -regularity of lattice X .

Notation F_β^α allows convenient computations for exponents and products of weights. The following property is immediate from the definitions.

Proposition 13. Suppose that $\alpha, \beta, \gamma \geq 0$ and $w \in F_\beta^\alpha$. Then $w^\gamma \in F_{\gamma\beta}^{\gamma\alpha}$ with the same constants. If $w > 0$ almost everywhere then $w^{-\gamma} \in F_{\gamma\alpha}^{\gamma\beta}$ with

the same constants. If a lattice X is F_β^α -regular and $\gamma > 0$ then lattice X^γ is $F_{\gamma\beta}^{\gamma\alpha}$ -regular with the same constants.

Proposition 14. *Suppose that $\alpha_0, \alpha_1, \beta_0, \beta_1 \geq 0$, $w_0 \in F_{\beta_0}^{\alpha_0}$ and $w_1 \in F_{\beta_1}^{\alpha_1}$. then $w_0 w_1 \in F_{\beta_0+\beta_1}^{\alpha_0+\alpha_1}$ with the appropriate estimates on the constants. Likewise, if X and Y are some lattices on $(S \times \Omega, \nu \times \mu)$ such that X is $F_{\beta_0}^{\alpha_0}$ -regular and Y is $F_{\beta_1}^{\alpha_1}$ -regular then lattice XY is $F_{\beta_0+\beta_1}^{\alpha_0+\alpha_1}$ -regular with the appropriate estimates on the constants.*

Indeed, since the sets of A_1 weights with constant at most C are logarithmically convex (see (8) below), it is easy to see that if $w_0 = \frac{\omega_{00}^{\alpha_0}}{\omega_{01}^{\beta_0}} \in F_{\beta_0}^{\alpha_0}$ and $w_1 = \frac{\omega_{10}^{\alpha_1}}{\omega_{11}^{\beta_1}} \in F_{\beta_1}^{\alpha_1}$ with some appropriate $\omega_{jk} \in A_1$ then

$$w_0 w_1 = \frac{\left(\frac{\alpha_0}{\omega_{00}^{\alpha_0+\alpha_1}} \omega_{10}^{\frac{\alpha_1}{\alpha_0+\alpha_1}} \right)^{\alpha_0+\alpha_1}}{\left(\omega_{01}^{\frac{\beta_0}{\beta_0+\beta_1}} \omega_{11}^{\frac{\beta_1}{\beta_0+\beta_1}} \right)^{\beta_0+\beta_1}} \in F_{\beta_0+\beta_1}^{\alpha_0+\alpha_1}$$

with an appropriate estimate for the constant.

It is remarkable that the statement of Proposition 14 can be reversed not only for weights but also for lattices. The following result is a generalization of [6, Theorem 2]; in the proof of Theorem 2 in Section 5 below it is applied with $\alpha_1 = 2$, $\alpha_0 = 1$, $\beta_0 = \beta_1 = 0$,

Theorem 15. *Suppose that X and Y are quasi-Banach lattices on $(S \times \Omega, \nu \times \mu)$ satisfying the Fatou property, XY is $F_{\beta_1}^{\alpha_1}$ -regular and Y is $F_{\beta_0}^{\alpha_0}$ -regular. Then lattice X is $F_{\beta_1+\alpha_0}^{\alpha_1+\beta_0}$ -regular.*

Examining the case of weighted $L_\infty(w)$ lattices with suitable weights shows that the conclusion of Theorem 15 is sharp in the sense that the indexes of regularity cannot be replaced by smaller values.

A complete proof of theorem 15 is given in Section 6 below. A weaker statement can be obtained directly from [6, Theorem 2]; however, the resulting indexes of regularity are too crude for our purposes. However, we may deduce the case needed in the present work from the following recently obtained result, which seems to be somewhat less involved technically than the proof of Theorem 15 in full generality that, among other things, uses a fixed point theorem.

Theorem 16 ([8, Theorem 14]). *Suppose that X is a Banach lattice of measurable functions on $S \times \Omega$ satisfying the Fatou property and $\alpha > 1$, $\beta > 0$. Then X is F_β^α -regular if and only if X' is $F_{\alpha-1}^{\beta+1}$ -regular.*

Indeed, suppose that under the conditions of Theorem 15 both lattices X and Y are r -convex with some $r > 0$ such that $\alpha_0 r > 1$ and $(\alpha_1 + \beta_0)r > 1$; these conditions are satisfied in the application to the proof of Theorem 2 in Section 5 below with some sufficiently close to 1 value of $r > 1$. Then lattice $(X^r)^{\frac{1}{2}}(Y^r)^{\frac{1}{2}}$ is $F_{\frac{\beta_1 r}{2}}^{\frac{\alpha_1 r}{2}}$ -regular and lattice Y^r is $F_{\beta_0 r}^{\alpha_0 r}$ -regular by Proposition 13, so lattice $(Y^r)'$ is $F_{\alpha_0 r - 1}^{\beta_0 r + 1}$ -regular by Theorem 16, thus lattice $(Y^r)^{\frac{1}{2}}$ is $F_{\frac{\alpha_0 r - 1}{2}}^{\frac{\beta_0 r + 1}{2}}$ -regular by Proposition 13. By the Lozanovsky factorization theorem [4] we have $L_1 = (Y^r)(Y^r)'$, and lattice $(X^r)^{\frac{1}{2}}L_1^{\frac{1}{2}} = (X^r)^{\frac{1}{2}}(Y^r)^{\frac{1}{2}}(Y^r)^{\frac{1}{2}}$ is $F_{\frac{(\beta_1 + \alpha_0)r + 1}{2}}^{\frac{(\alpha_1 + \beta_0)r + 1}{2}}$ -regular by Proposition 14, which by Theorem 16 implies that lattice $\left[(X^r)^{\frac{1}{2}}L_1^{\frac{1}{2}}\right]' = (X^r)^{\frac{1}{2}}$ is $F_{\frac{(\alpha_1 + \beta_0)r - 1}{2}}^{\frac{(\beta_1 + \alpha_0)r + 1}{2}}$ -regular, and thus lattice $(X^r)'$ is $F_{(\alpha_1 + \beta_0)r - 1}^{(\beta_1 + \alpha_0)r + 1}$ -regular by Proposition 13. Applying Theorem 16 to lattice $(X^r)'$ yields $F_{(\beta_1 + \alpha_0)r}^{(\alpha_1 + \beta_0)r}$ -regularity of lattice X^r , which by Proposition 13 implies the required $F_{\beta_1 + \alpha_0}^{\alpha_1 + \beta_0}$ -regularity of lattice X .

4. AN ESTIMATE FOR NONDEGENERATE OPERATORS

It is well known that if T is a nondegenerate operator in the sense of Definition 1 then the boundedness of T in $L_2\left(w^{-\frac{1}{2}}\right)$ implies that $w \in A_2$. However, in quantitative terms the standard argument establishing this (see, e. g., [10, Chapter 5, §4.6]) only yields an estimate $[w]_{A_2} \leq C\|T\|_{L_2(w^{-\frac{1}{2}}) \rightarrow L_2(w^{-\frac{1}{2}})}^4$, which is too rough for the proof of Theorem 2 in Section 5 below to work in full generality. The value $[w]_2$ cannot be estimated in terms of $C\|T\|_{L_2(w^{-\frac{1}{2}}) \rightarrow L_2(w^{-\frac{1}{2}})}$; see [2, §8.B].

Nevertheless, securing an additional restriction on the doubling constant of either the weight w or the weight w^{-1} leads to a suitable estimate. We denote by λ_n the Lebesgue measure on \mathbb{R}^n .

Proposition 17. *Suppose that T is a nondegenerate operator that is bounded in $L_2\left(w^{-\frac{1}{2}}\right)$ with a weight w on $(\mathbb{R}^n \times \Omega, \lambda_n \times \mu)$ such that either w or w^{-1} satisfies the doubling condition with a constant c_w . Then*

$$(3) \quad [w]_{A_2} \leq c_T C_w \|T\|_{L_2(w^{-\frac{1}{2}}) \rightarrow L_2(w^{-\frac{1}{2}})}^2$$

with a constant c_T independent of the weight w and a constant C_w depending only on c_w .

Indeed, let $m = \|T\|_{L_2(w^{-\frac{1}{2}}) \rightarrow L_2(w^{-\frac{1}{2}})}$ under the assumptions of Proposition 17. The argument in [6, Proposition 19] shows that

$$(4) \quad \int |Tf(\cdot)|^2 w(\cdot, \omega) \leq 2m^2 \int |f(\cdot)|^2 w(\cdot, \omega)$$

for almost all $\omega \in \Omega$ and all $f \in L_2(w^{-\frac{1}{2}}(\cdot, \omega))$.

Suppose that B is a ball in \mathbb{R}^n and let $B' = B + rx_0$ with $r > 0$ and $x_0 \in \mathbb{R}^n$ taken from the definition of a nondegenerate operator (Definition 1) as applied to T . It is easy to see that the boundedness of T implies that both $w(\cdot, \omega)$ and $w^{-1}(\cdot, \omega)$ are locally summable for almost all $\omega \in \Omega$. Substituting the condition (1) from the definition of a nondegenerate operator into (4), we see that

$$(5) \quad 2m^2 \int_B f^2(\cdot) w(\cdot, \omega) \geq \int |Tf(\cdot)|^2 w(\cdot, \omega) \geq \int_{B'} |Tf(\cdot)|^2 w(\cdot, \omega) \geq c^2 \left(\frac{1}{|B|} \int_B f \right)^2 \int_{B'} w(\cdot, \omega)$$

for almost all $\omega \in \Omega$ and all $f \in L_2(w^{-\frac{1}{2}}(\cdot, \omega))$ such that $f \geq 0$ and $\text{supp } f \subset B$. Putting $f = w^{-1}\chi_B$ into (5) yields

$$(6) \quad \left(\frac{1}{|B|} \int_{B'} w(\cdot, \omega) \right) \left(\frac{1}{|B|} \int_B w^{-1}(\cdot, \omega) \right) \leq 2c^{-2}m^2.$$

Since the balls $B = B(x, r)$ and $B' = B(x + rx_0, r)$ are mutually comparable in the sense that $B' \subset B(x, r(1 + |x_0|))$ and $B \subset B(x + rx_0, r(1 + |x_0|))$, the doubling condition of either the weight w or the weight w^{-1} implies that one of the factors on the left-hand side of (6) is suitably comparable to a similar factor with either B replaced by B' or vice versa. This observation yields (3), since both B and B' are arbitrary balls of \mathbb{R}^n .

We apply Proposition 17 to the situation arising in Theorem 3.

Proposition 18. *Suppose that Y is a Banach lattice on $(\mathbb{R}^n \times \Omega, \lambda_n \times \mu)$ with an order continuous norm, and let T be a nondegenerate operator acting boundedly in $Y^{\frac{1}{2}}$. Suppose also that lattice Y' is F_1^α -regular with some $\alpha > 0$. Then for every $f \in Y'$ there exists a majorant $w \geq |f|$, $\|w\|_{Y'} \leq m_2 \|f\|_{Y'}$, such that*

$$(7) \quad [w]_{A_2} \leq C_2 \|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}}^2$$

with some constants (C_2, m_2) independent of w and $\|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}}$.

To prove Proposition 18 we need to show that it is possible to take weights w in the conclusion of Theorem 3 that also satisfy $w \in F_1^\alpha$ with a

suitable control on the norm. To do this we adapt the fixed point argument from the proof of [7, Theorem 8]. This requires a few preparations.

We introduce the following sets of Muckenhoupt weights for $p > 1$:

$$(8) \quad \begin{aligned} BA_p^{(\text{MC})}(C) &= \{w \in A_p \mid [w]_{A_p} \leq C\}, \\ BA_1(C) &= \left\{w \in A_1 \mid \text{ess sup } \frac{Mw}{w} \leq C\right\}. \end{aligned}$$

Here “ BA_p ” denotes “the ball of A_p ”, and “(MC)” indicates that these sets are defined by the Muckenhoupt condition to avoid confusion with earlier work (e. g. [6, Section 3]), where different (for $p > 1$) sets $BA_p(C)$ were used. The latter have the advantage of being convex and they can also be used to establish the results of the present work; however, we do not need the convexity, and the basic facts about sets $BA_p^{(\text{MC})}(C)$ seem to be simpler. Such a definition is more in line with the rest of the arguments.

Proposition 19. *Sets $BA_p^{(\text{MC})}(C)$ are logarithmically convex and closed with respect to the convergence in measure.*

Indeed, the logarithmic convexity follows at once from the Hölder inequality, and the closedness with respect to the convergence in measure is obtained by twice applying the Fatou lemma: if $w_n \in BA_p^{(\text{MC})}(C)$ and $w_n \rightarrow w$ almost everywhere then

$$\begin{aligned} \frac{1}{\nu(B)} \int_B w(\cdot, \omega) &\leq \liminf_n \frac{1}{\nu(B)} \int_B w_n(\cdot, \omega) \leq \\ &C \liminf_n \left(\frac{1}{\nu(B)} \int_B w_n^{-\frac{1}{p-1}}(\cdot, \omega) \right)^{-(p-1)} = \\ &C \left(\limsup_n \frac{1}{\nu(B)} \int_B w_n^{-\frac{1}{p-1}}(\cdot, \omega) \right)^{-(p-1)} \leq \\ &C \left(\liminf_n \frac{1}{\nu(B)} \int_B w_n^{-\frac{1}{p-1}}(\cdot, \omega) \right)^{-(p-1)} \leq \\ &C \left(\frac{1}{\nu(B)} \int_B w^{-\frac{1}{p-1}}(\cdot, \omega) \right)^{-(p-1)} \end{aligned}$$

for all balls $B \subset S$ and almost all $\omega \in \Omega$, so $w \in BA_p^{(\text{MC})}(C)$.

According to Proposition 11, we can define for $\alpha > 0$, $\beta \geq 0$ the corresponding sets of F_β^α weights with a control on the constant by

$$BF_\beta^\alpha(C) = \left\{w^\alpha \mid w \in BA_{\frac{\beta}{\alpha}+1}^{(\text{MC})}(C)\right\},$$

$$BF_\beta^0(C) = \{w^{-\beta} \mid w \in BA_1(C), w > 0 \text{ almost everywhere}\}.$$

Consequently, these sets are also logarithmically convex and closed with respect to the convergence in measure.

Proposition 20. *Suppose that Z is a Banach¹ lattice on a σ -finite measurable space, $\omega_1 \in Z$, $\omega_1 > 0$ almost everywhere, $E \subset Z$ is a bounded set in Z such that $h \geq \omega_1$ for all $h \in E$. Then there exists some weight ω , $\omega > 0$ almost everywhere, such that $D = \{\log w \mid w \in E\}$ is a bounded set in $L_2(\omega^{-\frac{1}{2}})$.*

To prove Proposition 20, take any $a \in Z'$ such that $\|a\|_{Z'} = 1$ and $a > 0$ almost everywhere, any $\sigma \in L_1$ such that $\|\sigma\|_{L_1} = 1$ and $\sigma > 0$ almost everywhere, and define a weight $\omega = a \wedge \sigma(1 - [\log \omega_1]^-)^{-2}$. Then $\log w \in D$ implies

$$\begin{aligned} \int_{w \geq 1} |\log w|^2 \omega &\leq \int_{w \geq 1} |\log w|^2 a = \\ &\int_{w \geq 1} 4 \left| \log \left(w^{\frac{1}{2}} \right) \right|^2 a \leq 4 \int w a \leq 4 \|w\|_Z \|a\|_{Z'} \leq 4 \|w\|_Z \end{aligned}$$

and

$$\begin{aligned} \int_{w < 1} |\log w|^2 \omega &= \int_{w < 1} (-\log w)^2 \omega \leq \int (-\log \omega_1)^2 \omega = \\ &\int ([\log \omega_1]^-)^2 \omega \leq \int ([\log w]^-)^2 \sigma (1 - [\log \omega_1]^-)^{-2} \leq \int \sigma = 1, \end{aligned}$$

so indeed D is a bounded set in $L_2(\omega^{-\frac{1}{2}})$.

We now begin the proof of Proposition 18. For convenience, let $X = Y'$; lattice X always has the Fatou property. Let C be the constant from Theorem 3. We introduce a set

$$B_T = \left\{ w \in X \mid w \geq 0, \int |Tg|^2 w \leq \left(C \|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}} \right)^2 \int |g|^2 w \text{ for all } g \in L_2 \left(w^{-\frac{1}{2}} \right) \right\}.$$

Theorem 3 shows that this set is nonempty. By the complex interpolation B_T is logarithmically convex. The closedness of the set B_T with respect to the convergence in measure is verified routinely (see, e. g., the proof of [6, Proposition 16]): if $w_n \in B_T$ and $w_n \rightarrow w$ almost everywhere then we put $W = \sup_n w_n$ and see that by the Fatou lemma and the Lebesgue dominated

¹It is easy to see that Proposition 20 also holds true for quasi-normed lattices Z .

convergence theorem

$$(9) \quad \int |Tg|^2 w \leq \liminf_{j \rightarrow \infty} \int |Tg|^2 w_j \leq \left(C \|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}} \right)^2 \liminf_{j \rightarrow \infty} \int |g|^2 w_j \leq \\ \left(C \|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}} \right)^2 \lim_{j \rightarrow \infty} \int \left[\frac{w_j}{W} \right] |g|^2 W = \left(C \|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}} \right)^2 \int |g|^2 w$$

for all $g \in L_2 \left(W^{-\frac{1}{2}} \right)$, so extending (9) to all $g \in L_2 \left(w^{-\frac{1}{2}} \right)$ by density yields $w \in B_T$.

Suppose that $f \in X$. We may assume that $\|f\|_X = 1$ and $f > 0$ almost everywhere. By the assumptions lattice X is F_1^α -regular with some constants (C_1, m_1) . Let $0 < \beta \leq 1$ be a sufficiently small number to be determined later. We introduce a set $D = \{\log w \mid w \in X, w \geq \beta f, \|w\|_X \leq 1\}$ and a set-valued map $\Phi : D \times D \rightarrow 2^{D \times D}$ by

$$\Phi((\log u, \log v)) = \{(\log u_1, \log v_1) \in D \times D \mid \\ u_1 \in X, v_1 \in X, \|u_1\|_X \leq 1, \|v_1\|_X \leq 1, \\ u_1 \in B_T, v_1 \in BF_1^\alpha(C_1), f \vee u \vee v \leq A(u_1 \wedge v_1)\}$$

for all $(\log u, \log v) \in D \times D$ with a sufficiently large constant A to be determined in a moment.

Let $(\log u, \log v) \in D \times D$. Then $w = f \vee u \vee v \in X$ with $\|w\|_X \leq 3$. Applying Theorem 3 to function w yields a majorant $u_2 \in X$, $u_2 \geq w$, $\|u_2\|_X \leq 2\|w\|_X \leq 6$ such that $u_2 \in B_T$. On the other hand, by the F_1^α -regularity of X there exists some majorant $v_2 \in X$, $v_2 \geq w$, $\|v_2\|_X \leq m_1\|w\|_X \leq 3m_1$ such that $v_2 \in BF_1^\alpha(C_1)$. Setting $u_1 = \frac{1}{6}u_2$, $v_1 = \frac{1}{3m_1}v_2$ and choosing $A = 6 \vee 3m_1$ and $\beta = \frac{1}{A}$ shows that $(\log u_1, \log v_1) \in \Phi((\log u, \log v))$, so Φ takes nonempty values.

Now it suffices to establish that map Φ has a fixed point $(\log u, \log v) \in D \times D$, $\Phi((\log u, \log v)) \ni (\log u, \log v)$. If this is the case then $f \vee u \vee v \leq A(u \wedge v)$, so $w = A(u \vee v)$ is a majorant of f such that $\|w\|_X \leq 2A$ and w is pointwise equivalent to both u and v with constant A , which implies that $\|T\|_{L_2(w^{-\frac{1}{2}}) \rightarrow L_2(w^{-\frac{1}{2}})} \leq A^2 C \|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}}$ and $w \in F_1^\alpha$ with a constant depending only on A , C_1 and m_1 . Thus $w^{-1} \in F_\alpha^1 = A_{\alpha+1}$ by Proposition 13, and hence w^{-1} is a doubling weight with an estimate for the doubling constant depending only on A , C_1 and m_1 . Finally, Proposition 17 yields the required estimate 7 with a suitable constant C_2 .

Thus it suffices to verify that Φ satisfies the assumptions of the Fan-Kakutani fixed point theorem [1]. We apply Proposition 20, which gives a weight ω such that D is a bounded set in $L_2 \left(\omega^{-\frac{1}{2}} \right)$. We endow D with the

weak topology of $L_2\left(\omega^{-\frac{1}{2}}\right)$. D is a convex set that is closed with respect to the convergence in measure, and hence D is compact. Likewise, the graph Γ of Φ is a convex set, so it suffices to show that Γ is closed in the strong topology of $D \times D \times D \times D$, which easily follows from the closedness of Γ with respect to the convergence in measure by the Fatou property of the lattice X . This concludes the proof of Proposition 18.

5. PROOF OF THE MAIN RESULT

We begin by stating a recently developed (see [3]) quantitative estimate for the reverse Hölder inequality as it applies to A_p -regularity.

The Fujii-Wilson constant of a weight $w \in A_\infty$ on $S \times \Omega$, which gives an equivalent definition of the class A_∞ , is

$$[w]_{A_\infty} = \operatorname{ess\,sup}_{\omega \in \Omega} \sup_B \frac{\int_B M[\chi_B w](\cdot, \omega)}{\int_B w(\cdot, \omega)},$$

where the supremum is taken over all balls $B \subset S$. This constant is dominated by the Muckenhoupt constant $[w]_{A_p}$ for any p (see, e. g., [2, Proposition 2.2]). By [3, Theorem 1.1] any weight $w \in A_\infty$ satisfies the reverse Hölder inequality with all exponents $1 \leq r \leq 1 + \frac{1}{c[w]_{A_\infty}}$ for some constant c depending only on the properties of the underlying space (S, ν) , i. e.

$$\left(\frac{1}{\nu(B)} \int w^r(\cdot, \omega) \right)^{\frac{1}{r}} \leq C \frac{1}{\nu(B)} \int w(\cdot, \omega)$$

for almost all $\omega \in \Omega$ and all balls $B \subset S$ with some constant C independent of B .

If $w \in A_p$ then $w^{-\frac{1}{p-1}} \in A_{p'}$ and $[w]_{A_p} = \left[w^{-\frac{1}{p-1}} \right]_{A_{p'}}$, so it is seen immediately that $w \in A_p$ implies $w^r \in A_p$ and $[w^r]_{A_p} \leq c_2 [w]_{A_p}$ for all $1 \leq r \leq 1 + \frac{1}{c_1 [w]_{A_p}}$ with some constants c_1 and c_2 independent of w . This implies the following observation.

Proposition 21. *Suppose that a quasi-Banach lattice X on $(S \times \Omega, \nu \times \mu)$ is A_∞ -regular with (Fujii-Wilson) constants (C_{A_∞}, m) , and X is A_p -regular with some $1 \leq p < \infty$. Then X^r is also A_p -regular for all $1 < r \leq 1 + \frac{1}{c C_{A_\infty}}$ with some constant c independent of C_{A_∞} .*

Proposition 22. *Suppose that a normed lattice X on $(S \times \Omega, \nu \times \mu)$ is A_p -regular. Then lattices $X^\theta L_1^{1-\theta}$ are also A_p -regular for all $0 < \theta < 1$.*

Indeed, let $r > 1$. We have $Z = X^\theta L_1^{1-\theta} = (X^r)^{\frac{\theta}{r}} L_t^{1-\frac{\theta}{r}}$ with $t = \frac{1-\theta}{1-\frac{\theta}{r}} > 1$ and X^r is A_p -regular for small enough values of r by Proposition 21, which implies (by, e. g., Propositions 14 and 13) that Z is also A_p -regular.

We are now ready to prove implication $3 \Rightarrow 1$ of Theorem 2. Suppose that under the conditions of Theorem 2 operator T is bounded in X ; we need to show that lattices X and X' are A_1 -regular. By Proposition 8 it is sufficient to show that both X and X' are A_2 -regular.

The Fatou property together with p -convexity and q -concavity assumptions on X imply that both X and X' have order continuous norm (since, for example, $X' = (X^p)'^{\frac{1}{p}} L_1^{\frac{1}{p'}}$ and the product of a couple of Banach lattices has order continuous norm if one of the lattices has it), so $L_2 \cap X$ is dense in X and it is easy to see that T is bounded in X if and only if T^* is bounded in X' . Thus by the symmetry it suffices to prove that X' is A_2 -regular.

Let $1 < r, s \leq 2$ and $Y = X^r L_{s'} = (X^{rs})^{\frac{1}{s}} L_1^{1-\frac{1}{s}}$. Since X is p -convex, Y is a Banach lattice for all r and s satisfying $rs \leq p$. For clarity we may assume that $p \leq 2$. Let us fix $r = \frac{1+p}{2} < p$; then Y is p_1 -convex with $p_1 = \left(\frac{r}{p} + \frac{1}{s'}\right)^{-1} = \left(\frac{p+1}{2p} + \frac{1}{s'}\right)^{-1}$, so further restricting $s \leq \frac{4p}{3p+1}$ yields estimates $\frac{1}{s'} \leq \frac{p-1}{4p}$ and $1 < \frac{4p}{3p+1} = \left(\frac{p+1}{2p} + \frac{p-1}{4p}\right)^{-1} \leq p_1$. Thus lattice Y is also p_Y -convex with $p_Y = \frac{4p}{3p+1}$ for all $s \leq \frac{4p}{3p+1} = p_Y$.

We have

$$Y^{\frac{1}{2}} = X^{\frac{r}{2}} L_{s'}^{\frac{1}{2}} = X^{\frac{r}{2}} L_1^{\frac{1}{2s'}} = X^{\frac{r}{2}} \left(L_1^{\frac{1}{2s'(1-\frac{r}{2})}} \right)^{1-\frac{r}{2}} = X^{\frac{r}{2}} (L_{(2-r)s'})^{1-\frac{r}{2}},$$

and by the complex interpolation we see that

$$(10) \quad \|T\|_{Y^{\frac{1}{2}}} \leq \|T\|_X^{\frac{r}{2}} \|T\|_{L_{(2-r)s'}}^{1-\frac{r}{2}} \leq c(s')^{1-\frac{r}{2}}$$

with some constant c independent of s , since $\|T\|_{L_t} = O(t)$ as $t \rightarrow \infty$ for a Calderón-Zygmund operator T .

A similar computation shows that

$$(11) \quad Y^\varepsilon L_t^{1-\varepsilon} = X^{r\varepsilon} L_u^{1-r\varepsilon}$$

if $\frac{1}{u}(1-r\varepsilon) = \varepsilon \frac{1}{s'} + (1-\varepsilon) \frac{1}{t}$ for some $1 \leq u, t < \infty$ and $0 < \varepsilon < 1$. We choose $\varepsilon = \frac{1}{2}$ and $t = p'_Y$. Then (11) holds true with $u = \frac{2-r}{\frac{1}{s'} + \frac{1}{t}} = \frac{3-p}{2(\frac{1}{s'} + \frac{1}{t})}$. By making p smaller if necessary we may further assume that $t = p'_Y \geq \frac{8}{3-p}$, and for all $s \leq p_Y = t'$, we have $\frac{1}{s'} \leq \frac{1}{t}$ and $2 \leq t \frac{3-p}{4} = \frac{3-p}{2(\frac{1}{t} + \frac{1}{t})} \leq u \leq \frac{3-p}{2\frac{1}{t}} = t \frac{3-p}{2} \leq t$. Thus T is bounded in L_u uniformly in $1 \leq s \leq p_Y$ for the chosen values of ε and t . The complex interpolation yields

$$\|T\|_{Y^\varepsilon L_t^{1-\varepsilon}} \leq \|T\|_X^{r\varepsilon} \|T\|_{L_u}^{1-r\varepsilon} \leq c_1$$

with a constant c_1 independent of $1 \leq s \leq p_Y$.

Lattice $Y^\varepsilon L_t^{1-\varepsilon} = Y^{\frac{1}{2}} L_t^{\frac{1}{2}}$ is p_2 -convex with $p_2 = \left(\varepsilon \frac{1}{p_Y} + (1-\varepsilon) \frac{1}{t}\right)^{-1} = 2 \left(\frac{1}{p_Y} + 1 - \frac{1}{t}\right)^{-1} = 2$, so we may apply Theorem 4 to it. This shows that $\left(Y^{\frac{1}{2}} L_t^{\frac{1}{2}}\right)' = Y'^{\frac{1}{2}} L_{t'}^{\frac{1}{2}}$ is A_1 -regular, or F_0^1 -regular in terms of Definition 10.

By Proposition 13 lattice $Y' L_{t'} = \left(Y'^{\frac{1}{2}} L_{t'}^{\frac{1}{2}}\right)^2$ is F_0^2 -regular. Lattice $L_{t'}$ is A_1 -regular, or F_0^1 -regular. Therefore by Theorem 15 lattice Y' is F_1^2 -regular.

Thus (10) by Proposition 18 implies that lattice Y' is A_2 -regular with constants (C_3, m_3) satisfying $C_3 \leq c_3 \|T\|_{Y^{\frac{1}{2}} \rightarrow Y^{\frac{1}{2}}}^2 \leq c_3 c^2(s')^{2-r}$ for some c_3 and m_3 independent of s . By Proposition 21 lattice $(Y')^\rho$ is then A_2 -regular for all $1 \leq \rho \leq 1 + \frac{1}{c_4(s')^{2-r}}$ with a constant c_4 independent of s .

Observe that $Y' = \left[(X^{rs})^{\frac{1}{s}} L_1^{1-\frac{1}{s}}\right]' = (X^{rs})'^{\frac{1}{s}}$ and $(Y')^\rho = (X^{rs})'^{\frac{\rho}{s}}$. Setting $\rho = 1 + \frac{1}{c_4(s')^{2-r}}$ yields

$$\frac{\rho}{s} = \frac{c_4(s')^{2-r} + 1}{c_4(s')^{2-r}} \cdot \frac{s' - 1}{s'} = \frac{c_4(s')^{3-r} - c_4(s')^{2-r} + s' - 1}{c_4(s')^{3-r}}.$$

Since $0 < 2-r < 1$, we have $c_4(s')^{2-r} \leq s' - 1$ for sufficiently large values of s' , that is, for sufficiently close to 1 values of s , so we have $\frac{\rho}{s} > 1$ and $\frac{s}{\rho} < 1$ for small enough values of s . We fix such an s . Lattice $[(Y')^\rho]^{\frac{s}{\rho}} = (X^{rs})'$ is A_2 -regular, and by Proposition 22 lattice $X' = \left[(X^{rs})^{\frac{1}{rs}}\right]' = (X^{rs})'^{\frac{1}{rs}} L_1^{1-\frac{1}{rs}}$ is also A_2 -regular. This concludes the proof of Theorem 2.

6. PROOF OF THEOREM 15

Compared to [6, Theorem 2], the proof of Theorem 15 essentially requires only minor technical adjustments; however, to avoid confusion we provide a complete version of it. The only apparent difficulty that arises in direct translation of the proof is that the sets of the corresponding F_β^α -majorants seem to lack convexity for $\alpha \neq 1$; however, they are still logarithmically convex, which suffices to establish closedness of the graph of the map using the same method. We also use a different ambient space for the map, which makes approximating the problem by restricting the conditions to sets of finite measure unnecessary. This modification also allows us to avoid using a compactness-type result for sets closed with respect to the convergence in measure, since the standard weak compactness of sets in a weighted L_2 space suffices.

See Section 4 above for the definition of sets $BF_\beta^\alpha(C)$.

Proposition 23. *Suppose that $u, v \in BF_\beta^\alpha(C)$ with a constant C . Then $u \vee v \in BF_\beta^\alpha(2C)$. If $\alpha = 0$ then $u \vee v \in BF_\beta^\alpha(C)$.*

Indeed, according to Proposition 11, if $\alpha > 0$ it suffices to prove Proposition 23 for the corresponding sets $BA_p^{(\text{MC})}(C)$ in place of $BF_\beta^\alpha(C)$. We have

$$\begin{aligned} \frac{1}{\nu(B)} \int_B (u \vee v)(\cdot, \omega) &\leq \frac{1}{\nu(B)} \int_B u(\cdot, \omega) + \frac{1}{\nu(B)} \int_B v(\cdot, \omega) \leq \\ C \left[\left(\frac{1}{\nu(B)} \int_B u^{-\frac{1}{p-1}}(\cdot, \omega) \right)^{-(p-1)} + \left(\frac{1}{\nu(B)} \int_B v^{-\frac{1}{p-1}}(\cdot, \omega) \right)^{-(p-1)} \right] &\leq \\ 2C \left(\frac{1}{\nu(B)} \int_B (u \vee v)^{-\frac{1}{p-1}}(\cdot, \omega) \right)^{-(p-1)} & \end{aligned}$$

for all balls $B \subset S$ and almost all $\omega \in \Omega$, so indeed $u \vee v \in BA_p^{(\text{MC})}(C)$.

In the case $\alpha = 0$ it suffices to show that $u, v \in BA_1(C)$ implies $u \wedge v \in BA_1(C)$, which follows at once from the estimates $M(u \wedge v) \leq Mu \leq Cu$ and $M(u \wedge v) \leq Mv \leq Cv$.

We begin the proof of Theorem 15. First of all, since for all $\delta > 0$ the statement of Theorem 15 for lattices X and Y is equivalent to the same statement for lattices X^δ and Y^δ with all indices multiplied by δ , and since for any quasi-Banach lattice Z lattice Z^δ is (up to a renorming) Banach for small enough values of δ (see, e. g., [9, Theorem 3.2.1]), we may assume that lattices XY , X and Y are all Banach.

Suppose that lattice XY is $F_{\beta_1}^{\alpha_1}$ -regular with constants (C_{XY}, m_{XY}) and Y is $F_{\beta_0}^{\alpha_0}$ -regular with constants (C_Y, m_Y) . We can choose C large enough (depending on C_{XY} and C_Y) that the $F_{\beta_1}^{\alpha_1}$ -majorants in XY lie in $BF_{\beta_1}^{\alpha_1}(C)$ and the $F_{\beta_0}^{\alpha_0}$ -majorants in Y belong to $BF_{\beta_0}^{\alpha_0}(C)$.

Take any $\omega_0 \in Y$ such that $\|\omega_0\|_Y > 0$. There exists an F_β^α majorant $\omega_1 \in BF_\beta^\alpha(C)$ for ω_0 . We may assume that $\|\omega_1\|_Y = 1$. Let

$$D = \{ \log w \mid w \in BF_\beta^\alpha(2C), w \geq \omega_1, \|w\|_Y \leq 2 \}.$$

Suppose that $f \in X$; we need to prove that there exists a suitable $F_{\beta_1+\alpha_0}^{\alpha_1+\beta_0}$ -majorant for f . We may assume that $f > 0$ almost everywhere and that $\|f\|_X = 1$.

Take any function $\log w \in D$. Then $fw \in XY$ with norm at most 2, and there exist some majorants $g \geq f$, $g \in BF_{\beta_1}^{\alpha_1}(C)$, $\|g\|_{XY} \leq 2m_{XY}$. It is easy to see that (see, e. g., [6, (16)])

$$\|g\|_{XY} \geq (1 + \|\omega_0\|_Y)^{-1} \inf_{\substack{\|b\|_Y \leq 1 + \|\omega_0\|_Y, \\ b \geq \omega_0}} \|gb^{-1}\|_X,$$

so there exists some $b \in Y$, $b \geq \omega_0$, $\|b\|_Y \leq 2$ such that $\|gb^{-1}\|_X \leq 4m_{XY}$. Now let $v \geq b$, $v \in BF_{\beta_0}^{\alpha_0}(C)$, $\|v\|_Y \leq 2m_Y$ be an $F_{\beta_0}^{\alpha_0}$ -majorant for b , and

let $w_1 = \left(\frac{1}{2m_Y}v\right) \vee \omega_1$. Then $\|gw_1^{-1}\|_X \leq 2m_Y \|gv^{-1}\|_X \leq 2m_Y \|gb^{-1}\|_X \leq 8m_Y m_{XY}$, $\|w_1\|_Y \leq 2$ and $w_1 \in BF_{\beta_0}^{\alpha_0}(2C)$ by Proposition 23. This shows that a set-valued map $\Phi : D \rightarrow 2^D$ defined by

$$\Phi(\log w) = \{\log w_1 \in D \mid g \geq fw, g \in BF_{\beta_1}^{\alpha_1}(C), \|gw_1^{-1}\|_X \leq 8m_Y m_{XY}\}$$

takes nonempty values.

If map Φ has a fixed point $\log w \in D$, $\Phi(\log w) \ni \log w$ then there exists some function $g \geq fw$, $g \in BF_{\beta_1}^{\alpha_1}(C)$ such that $\|gw^{-1}\|_X \leq 8m_Y m_{XY}$ and $f_1 = gw^{-1} \in F_{\beta_1+\alpha_0}^{\alpha_1+\beta_0}$ with a suitable estimate of the constant by Propositions 13 and 14, so f_1 is then a suitable $F_{\beta_1+\alpha_0}^{\alpha_1+\beta_0}$ -majorant for f .

Thus it suffices to show that Φ satisfies the conditions of the Fan-Kakutani fixed point theorem [1]: that D is a compact set in a locally convex linear topological space such that Φ has closed graph and that Φ takes convex closed values that are compact.

By Proposition 20 there exists a weight ω such that D is a bounded set in $L_2(\omega^{-\frac{1}{2}})$. We endow D with the weak topology of $L_2(\omega^{-\frac{1}{2}})$. Since D is convex and closed with respect to the convergence in measure, D is a closed and bounded convex set in $L_2(\omega^{-\frac{1}{2}})$; hence D is a compact set.

It is easy to see that the graph Γ of Φ is a convex set, so it suffices to show that Γ is a closed set in the strong topology of the ambient space $L_2(\omega^{-\frac{1}{2}})$.

Suppose that $\log a_j, \log u_j \in D$, $\log a_j \in \Phi(\log u_j)$, $\log a_j \rightarrow \log A \in D$ and $\log u_j \rightarrow \log U \in D$ in $L_2(\omega^{-\frac{1}{2}})$; we need to verify that $\log A \in \Phi(\log U)$.

By passing to a subsequence we may assume that we also have $\log a_j \rightarrow \log A$ and $\log u_j \rightarrow \log U$ in the sense of the convergence almost everywhere. We form a nonincreasing sequence $\log \alpha_j = \bigvee_{k \geq j} \log a_k \geq \log a_j$ and a non-decreasing sequence $\log \eta_j = \bigwedge_{k \geq j} \log u_k \leq \log u_j$ of measurable functions such that $\log \alpha_j \rightarrow \log A$ and $\log \eta_j \rightarrow \log U$ almost everywhere.

Condition $\log a_j \in \Phi(\log u_j)$ implies that sets

$$W_j = \left\{ \log g \mid g \geq f\eta_j, g \in BF_{\beta_1}^{\alpha_1}(C), \|g\alpha_j^{-1}\|_X \leq 8m_Y m_{XY} \right\} \supset \left\{ \log g \mid g \geq fu_j, g \in BF_{\beta_1}^{\alpha_1}(C), \|ga_j^{-1}\|_X \leq 8m_Y m_{XY} \right\}$$

are nonempty, and W_j is a nonincreasing sequence of sets. Since for all $\log g \in W_1$ we have $g \geq f\omega_1 > 0$ almost everywhere and functions g are uniformly bounded in the weighted Banach lattice $X(\alpha_1)$, by Proposition 20 there exists a weight ω_2 such that W_1 is a bounded set in $L_2(\omega_2^{-\frac{1}{2}})$. It is easy to see that the sets W_j are convex and closed with respect to the convergence

in measure (and thus also in the strong topology of lattices satisfying the Fatou property), so they are compact in the weak topology of $L_2\left(\omega_2^{-\frac{1}{2}}\right)$. This implies that the set $\bigcap_j W_j$ is nonempty, and so there exists some function $g \in BF_{\beta_1}^{\alpha_1}(C)$ such that $g \geq f\eta_j$ and $\|g\alpha_j^{-1}\|_X \leq 8m_Y m_{XY}$ for all j . Thus $g \geq f\left(\bigvee_j \eta_j\right) = fU$ and $\|ga^{-1}\|_X \leq \bigvee_j \|g\alpha_j^{-1}\|_X \leq 8m_Y m_{XY}$ by the Fatou property. The existence of such a function g implies that $\log A \in \Phi(\log U)$ as claimed, which concludes the proof of Theorem 15.

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